TRANSVERSE VIBRATIONS OF THE BELTS

A common phenomenon in a system with driven belt transmission is the slip between belt and pulley. This, generally, is due to insufficient friction or due to centrifugal forces and sometimes to vibrations in the belt. In our system, some delays in velocity between rotor and motor pulley were imputed to the centrifugal effects. After a brief analysis, the phenomenon appeared more complex than expected and the delay measured at different spinning speeds was linked to a certain type of belt vibration. In the following section a complete analysis of the driven belt transmission will be shown with a study of the transverse vibrations of the moving belt. Some experimental results will be shown in comparison with the numerical predictions.

Analysis of the transmission

In this document a classical analysis of the driven belt transmission will be done in order to complete the argument. For the static analysis of the belt, we study an infinitesimal portion of it considering the contributes of tensions in the belt, friction forces and centrifugal forces. In the figure 34, forces acting on the portion are indicated with their names.

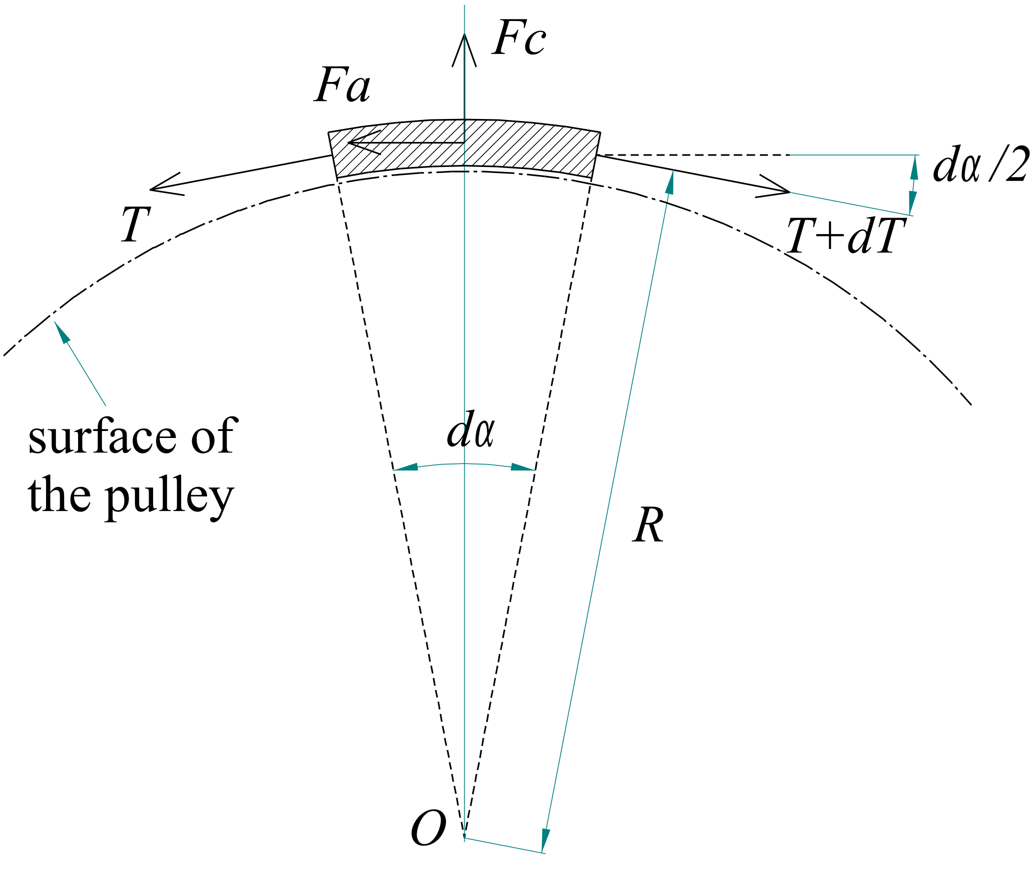


Figure 34

For the equilibrium of the belt portion in tangential and horizontal direction, we obtain

$$\begin{cases} (T + dT - T)\cos\dfrac{d\alpha}{2} - F_a = 0 \\ (T + dT + T)\sin\dfrac{d\alpha}{2} - F_c - pRd\alpha = 0 \end{cases} . \tag{1}$$

By using the second equation, the condition to avoid separation between belt and pulley comes from the condition the specific pressure has to be positive or, as extreme condition

$$(T + dT + T)\sin\frac{d\alpha}{2} \geq F_c . \tag{2}$$

If contact between belt and pulley is guaranteed, the condition to avoid slip comes from the first equation of system (1), and it states the friction force has to be higher than the tension in the belt:

$$(T + dT - T)\cos\frac{d\alpha}{2} \leq F_a . \tag{3}$$

Now, because the angle $d\alpha$ is infinitesimal, the conditions in the equations (2) and (3) become

$$\begin{cases} dT \leq F_a \\ Td\alpha \geq F_c \end{cases} . \tag{4}$$

To solve the problem it's necessary to express the friction and centrifugal forces acting on the portion of the belt in figure 34:

$$F_c = m\omega^2 R = (\rho Rd\alpha)\omega^2 R , \tag{5}$$

and

$$F_a = \mu\left[(T + dT + T)\sin\frac{d\alpha}{2} - F_c\right] = \mu\left(T - \rho R^2\omega^2\right)d\alpha . \tag{6}$$

By using the first equation of the system (4) and the equation (6) we have the following fundamental relation:

$$dT \leq \mu \left(T - \rho R^2 \omega^2\right) d\alpha$$

or

$$\frac{dT}{T - \rho R^2 \omega^2} \leq \mu \, d\alpha \ . \qquad (7)$$

By integrating the equation (7) between the extremes of the belt winded up to the pulley, we obtain:

$$\int_1^2 \frac{dT}{T - \rho R^2 \omega^2} \leq \int_0^\vartheta \mu \, d\alpha \ ,$$

$$\frac{T_2 - \rho R^2 \omega^2}{T_1 - \rho R^2 \omega^2} \leq e^{\mu \vartheta} \ , \qquad (8)$$

where ϑ is the winding angle of the belt on the motor pulley. The equation (8) gives the condition to avoid slip between belt and pulley.
It’s also possible to calculate the specific pressure of the belt on the pulley during working. To make this it’s necessary to form the hypothesis that the tension in the belt will have a linear distribution:

$$T(\alpha) = \frac{T_2 - T_1}{\vartheta} \alpha + T_1 \ . \qquad (9)$$

Therefore, by using the second equation of (1) we have the specific pressure

$$p(\alpha) = \frac{T - \rho \omega^2 R^2}{R} = \frac{T_2 - T_1}{R\vartheta} \alpha + \frac{T_1}{R} - \rho \omega^2 R \ . \qquad (10)$$

That means to avoid separation between belt and pulley in any point, it’s necessary to have in any angular position

$$p(\alpha) > 0 \ . \qquad (11)$$

Analysis of the vibration

In our machine, a slip could appear between the belt and the motor pulley where the winding angle is small and angular velocities are high. We will see in the following section, using the previous equations, that the level of the centrifugal forces in the motor pulley is not sufficient to produce a separation. We will also see that the delay of the rotor referred to the motor appears at different angular velocities and at not so high levels.
This suggests the presence of another phenomenon. In fact, the free portion of the belt has the possibility to vibrate as a string of a guitar. Mainly in the active side of the belt, where during spinning the tension is higher, it's possible to note transverse vibrations at different angular velocities (figure 35). At these spinning frequencies, a delay of the rotor referred to the motor is measurable and it increases with the speed (figure 42).
If the pulley of the driven rotor is slightly out of axis, it interacts with the free part of the belt as a finger does with a string of a guitar. In the case of the belt, the vibration will increase enormously when an harmonic of the rotation frequency of the rotor pulley excites a frequency of the belt.
Moreover, it's important to consider that in a translating string, the natural frequencies decrease with the velocity of translation.
It's possible to study the behavior of the belt during its translation by a mathematical analysis.

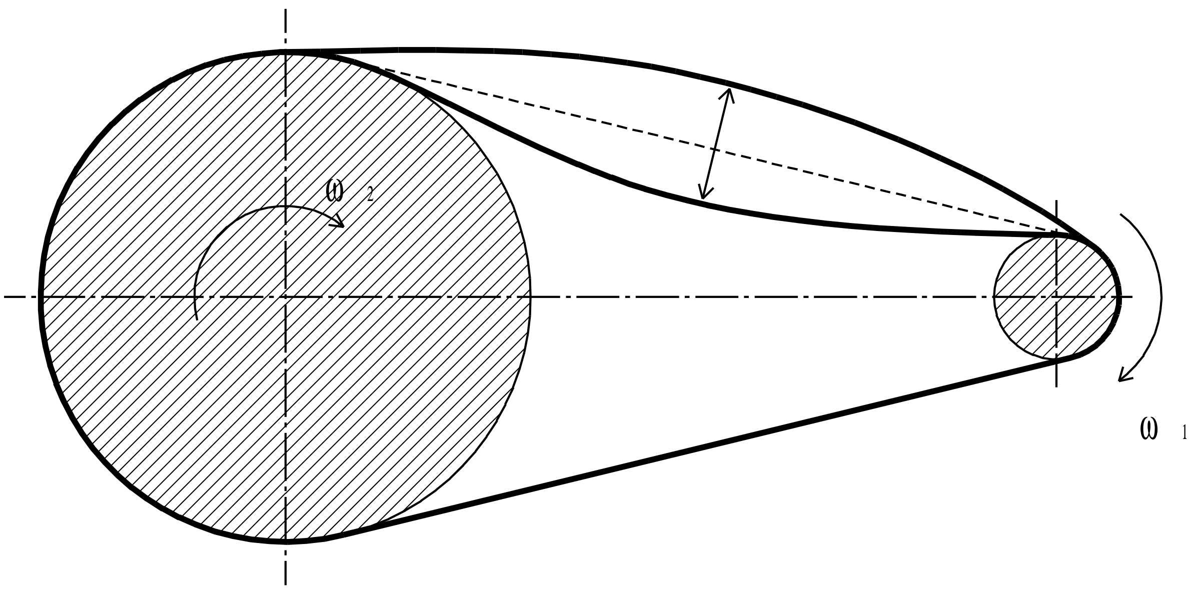


Figure 35

If we consider a small portion of belt dx (figure 36), it's possible to express forces acting on it and write down the equilibrium equation for the transverse motion.

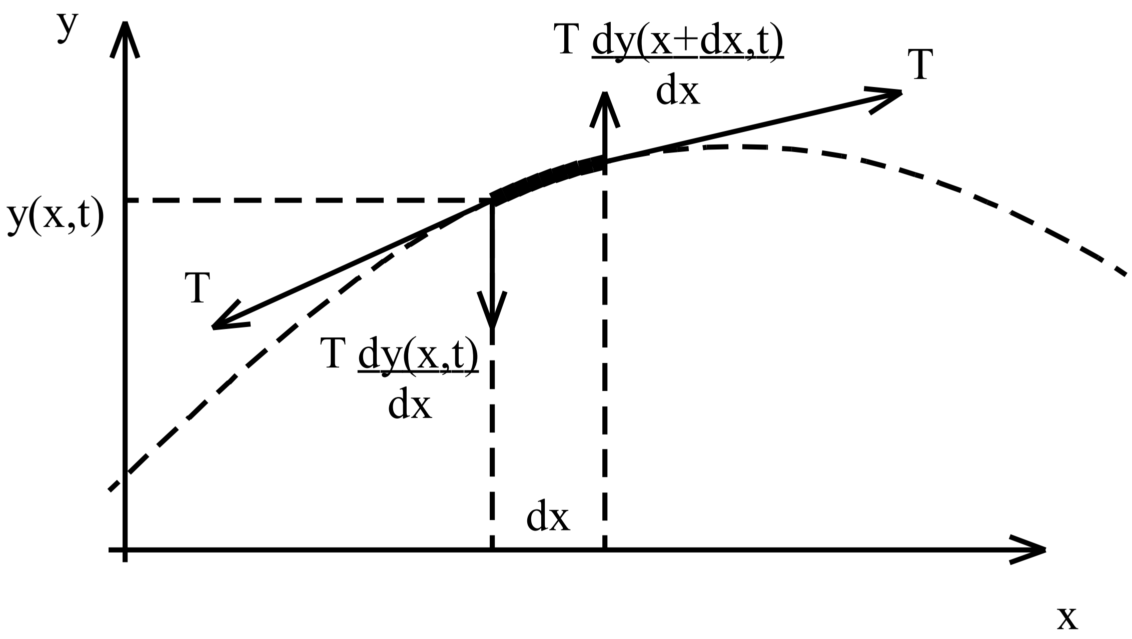


Figure 36

In order to obtain the right form of the equation for the transverse motion, it's necessary to take into account that the coordinate *x* changes with the time and its derivative is the translational velocity of the belt.
To write the equilibrium equation we have to express transverse forces due to the tension of the belt and inertial forces on the element *dx*.
The tension *T* is considered constant and the forces at the ends of the element *dx* are, as in figure 36,

$$F_x = T\frac{\partial}{\partial x}y(x(t),t)\,, \tag{12}$$

$$F_{x+dx} = T\frac{\partial}{\partial x}y\left(x(t)+dx,t\right)\,, \tag{13}$$

by using the differential calculus we know that

$$y(x+dx) = y(x) + \frac{d}{dx}y(x)dx + o(x) \tag{14}$$

and for the transverse force at the right end of the belt portion we obtain

$$F_{x+dx} = T\frac{\partial}{\partial x}y\left(x(t)+dx,t\right) = T\left(\frac{\partial}{\partial x}y\left(x(t),t\right)+\frac{\partial^2}{\partial x^2}y\left(x(t),t\right)dx\right). \tag{15}$$

By adding the forces in the equations (12) and (15) we obtain the resultant force in the vertical direction due to the tension of the belt:

$$F_v = T\frac{\partial^2}{\partial x^2}y(x(t),t)dx. \tag{16}$$

By using the linear density of the belt ρ we calculate the inertial force of the portion of the belt as

$$F_i = \rho\, dx\frac{d^2}{dt^2}y\left(x(t),t\right). \tag{17}$$

In order to explicate the inertial force in the equation (17) some passages of differential calculus are needed. Now, if $x = x(t)$, the calculus gives

$$\frac{d}{dt}\left[y(x,t)\right] = \frac{\partial y}{\partial x}\frac{dx}{dt}+\frac{\partial y}{\partial t}, \tag{18}$$

$$\frac{d}{dt}\left(\frac{\partial y}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)\frac{dx}{dt}+\frac{\partial^2 y}{\partial x\partial t}, \tag{19}$$

$$\frac{d}{dt}\left(\frac{\partial y}{\partial t}\right) = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial t}\right)\frac{dx}{dt}+\frac{\partial^2 y}{\partial t^2}, \tag{20}$$

$$\frac{d^2}{dt^2}\left[y(x,t)\right] = \frac{d}{dt}\left(\frac{\partial y}{\partial x}\right)\frac{dx}{dt}+\frac{\partial y}{\partial x}\frac{d^2x}{dt^2}+\frac{d}{dt}\left(\frac{\partial y}{\partial t}\right), \tag{21}$$

$$\frac{d^2}{dt^2}\left[y(x,t)\right] = \frac{\partial^2 y}{\partial x^2}\left(\frac{dx}{dt}\right)^2+2\frac{\partial^2 y}{\partial x\partial t}\frac{dx}{dt}+\frac{\partial y}{\partial x}\frac{d^2x}{dt^2}+\frac{\partial^2 y}{\partial t^2}. \tag{22}$$

Considering that the first derivative of x with respect to time is the translational velocity of the belt and the second derivative is the acceleration, we obtain

$$\frac{d^2}{dt^2}\left[y(x,t)\right] = V^2\frac{\partial^2 y}{\partial x^2}+2V\frac{\partial^2 y}{\partial x\partial t}+\dot{V}\frac{\partial y}{\partial x}+\frac{\partial^2 y}{\partial t^2}. \tag{23}$$

If we form the hypothesis the translational velocity *V* of the belt is constant, we can write the equation for an axially moving belt as

$$\frac{\partial^2}{\partial t^2} y(x,t) + 2V \frac{\partial^2}{\partial x \partial t} y(x,t) + \left(V^2 - \frac{T}{\rho} \right) \frac{\partial^2}{\partial x^2} y(x,t) = 0 . \tag{24}$$

The eigenvalue problem is obtained from the separable solution $y(x,t) = u(x)\, e^{\lambda t}$. By substituting the separable solution in the fundamental equation (24), we obtain

$$\lambda^2 u + \lambda\, 2V \frac{\partial u}{\partial x} + \left(V^2 - \frac{T}{\rho} \right) \frac{\partial^2 u}{\partial x^2} = 0 , \tag{25}$$

or

$$\left(V^2 - \frac{T}{\rho} \right) u'' + 2\lambda V u' + \lambda^2 u = 0 . \tag{26}$$

The eigenvalue problem for the differential equation (26) is obtained by using another separable solution $u(x) = Ue^{\gamma x}$. To find γ, after substituting the separable solution in (26), we have to solve the following equation:

$$\left(V^2 - \frac{T}{\rho} \right) \gamma^2 + 2\lambda V \gamma + \lambda^2 = 0 , \tag{27}$$

and the solutions are

$$\gamma_1 = \frac{\lambda_1}{V - \sqrt{\frac{T}{\rho}}} , \tag{28}$$

$$\gamma_2 = \frac{\lambda_1}{V + \sqrt{\frac{T}{\rho}}} , \tag{29}$$

$$\gamma_3 = \frac{\lambda_2}{V - \sqrt{\frac{T}{\rho}}} , \tag{30}$$

$$\gamma_4 = \frac{\lambda_2}{V + \sqrt{\frac{T}{\rho}}} . \tag{31}$$

Therefore, the general solution for *u* becomes

$$u(x) = U_1 e^{\gamma_1 x} + U_2 e^{\gamma_2 x} + U_3 e^{\gamma_3 x} + U_4 e^{\gamma_4 x} . \tag{32}$$

With $u(0) = u(L) = 0$ as boundary conditions we have the two equivalent systems of equations

$$\begin{cases} U_1 = -U_2 \\ e^{\frac{\lambda_1 L}{V - V_{CR}}} - e^{\frac{\lambda_1 L}{V + V_{CR}}} = 0 \end{cases} , \tag{33}$$

$$\begin{cases} U_3 = -U_4 \\ e^{\frac{\lambda_2 L}{V - V_{CR}}} - e^{\frac{\lambda_2 L}{V + V_{CR}}} = 0 \end{cases} \tag{34}$$

and from the second of each system

$$e^{\frac{2\lambda_1 V_{CR} L}{V^2 - V_{CR}{}^2}} = 1 , \tag{35}$$

$$e^{-\frac{2\lambda_2 V_{CR} L}{V^2 - V_{CR}{}^2}} = 1 , \tag{36}$$

$$\frac{\lambda_{1,2} L V_{CR}}{V^2 - V_{CR}{}^2} = \pm jn\pi . \tag{37}$$

The exact eigenvalues of the fundamental equation (24) are

$$\lambda_{1,2} = \pm j \frac{n\pi}{L} V_{CR} \left(1 - \frac{V^2}{V_{CR}{}^2} \right) = \pm j\omega_n \tag{38}$$

or

$$\lambda_{1,2} = \pm j \frac{n\pi}{L} \sqrt{\frac{T}{\rho}} \left(1 - V^2 \frac{\rho}{T} \right) = \pm j\omega_n , \tag{39}$$

with n = 1, 2, 3, ..., confirming that all eigenvalues vanish at $V = \sqrt{\frac{T}{\rho}}$. For a free translating string, this is the only critical speed and the only unstable speed. In fact the system stiffness operator vanishes for the critical speed, and any continuous function satisfying the boundary conditions is a critical speed eigenfuction.
In our study we are mainly interested to express the relation between the natural frequencies and the translational velocity *V*.
Finally, to write down the equation of the vibrating belt let's assume

$$\gamma_1 = \frac{j\omega_n}{V - \sqrt{\frac{T}{\rho}}} = j\alpha \quad , \tag{40}$$

$$\gamma_2 = \frac{j\omega_n}{V + \sqrt{\frac{T}{\rho}}} = j\beta \quad , \tag{41}$$

$$\gamma_3 = \frac{-j\omega_n}{V - \sqrt{\frac{T}{\rho}}} = -j\alpha \quad , \tag{42}$$

$$\gamma_4 = \frac{-j\omega_n}{V + \sqrt{\frac{T}{\rho}}} = -j\beta \quad , \tag{43}$$

in order to obtain

$$y(x,t) = U_1 e^{j(\alpha x + \beta t)} - U_1 e^{j(\alpha x + \beta t)} + U_3 e^{-j(\alpha x + \beta t)} - U_3 e^{-j(\alpha x + \beta t)} . \tag{44}$$

The boundary conditions are completely satisfied, so its possible to choose any value for U_1 and U_3 in order to obtain a real solution:

$$U_1 = U_3 = \frac{B_1}{2} , \tag{45}$$

and by substituting in equation (44) we obtain

$$y_1(x,t) = B_1 \cos(\alpha x + \omega_n t) - B_1 \cos(\beta x + \omega_n t) . \tag{46}$$

Moreover we can choose

$$U_1 = -U_3 = \frac{B_2}{2j}, \tag{47}$$

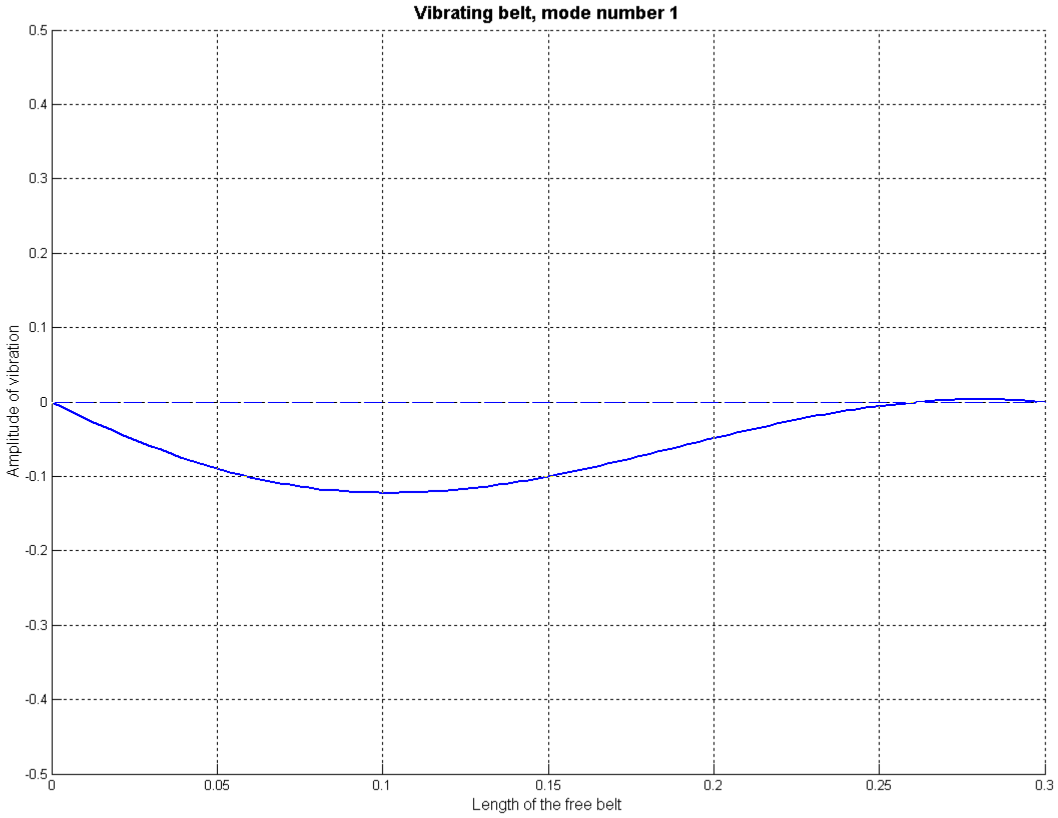
and by substituting again in the equation (44) we have

$$y_2(x,t) = B_2 \sin(\alpha x + \omega_n t) - B_2 \sin(\beta x + \omega_n t) . \tag{48}$$

The final function describing the behavior of the vibrating belt is

$$y(x,t) = B_1 \left(\cos(\alpha x + \omega_n t) - \cos(\beta x + \omega_n t) \right) + B_2 \left(\sin(\alpha x + \omega_n t) - \sin(\beta x + \omega_n t) \right) . \tag{49}$$

The first three modal shapes of the transverse vibration of the belt simulated by the equation (8,49) are reported in the figures 37, 38 and 39.
As it's possible to observe, the points of the belt don't cross the zero configuration at the same time.



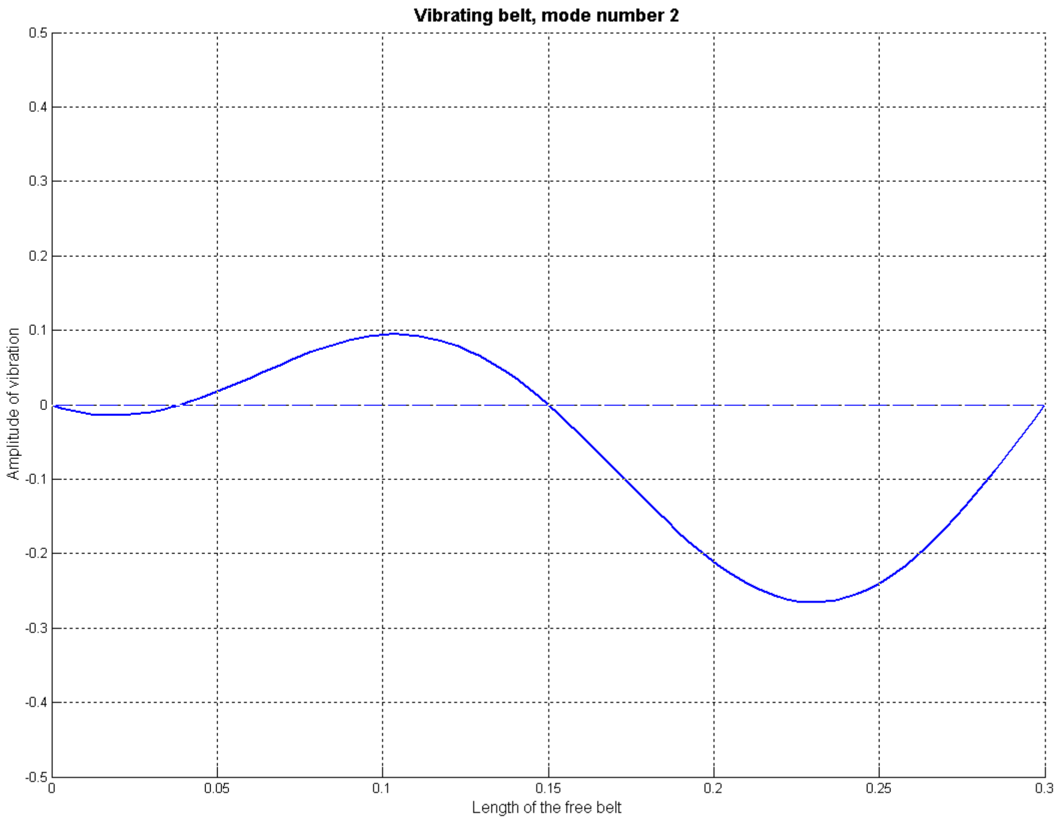


Figure 37 and 38

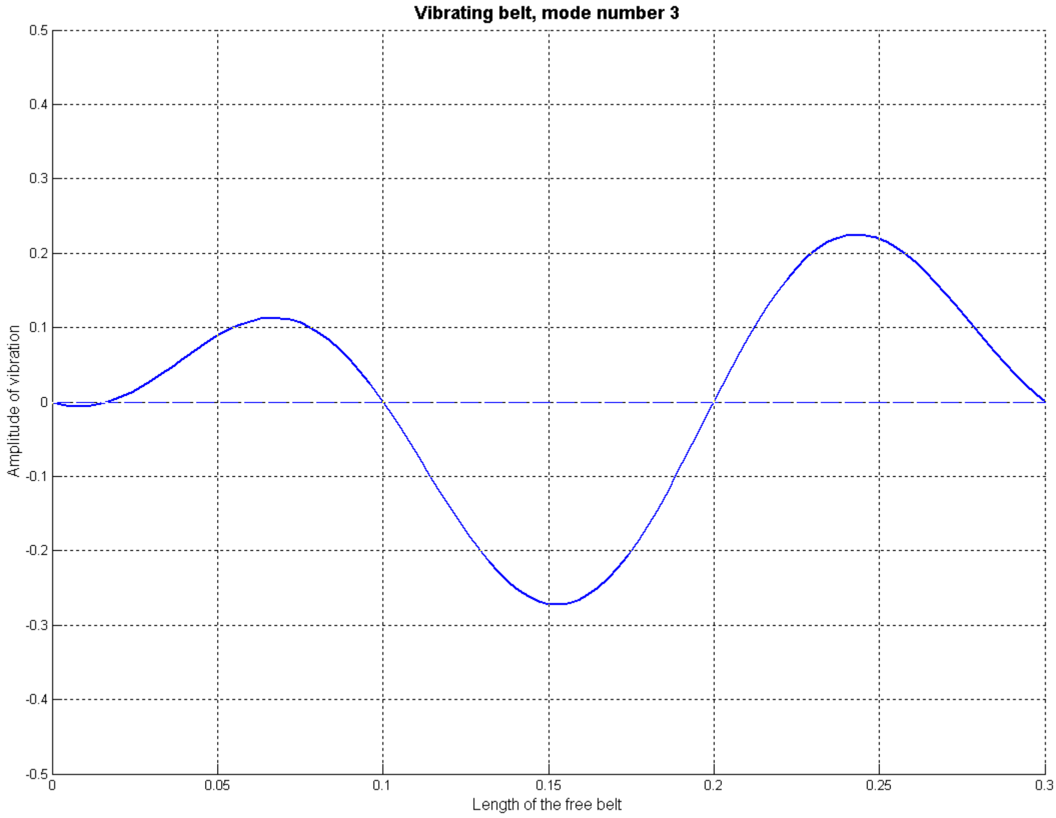

Vibrating belt, mode number 3
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
Amplitude of vibration
0
0.05
0.1
0.15
0.2
0.25
0.3
Length of the free belt

Figure 39

Slip between belt and motor pulley due to the vibration of the belt

Before to consider the transverse vibration of the belt, it's useful to analyze the forces and the specific pressure. The values measured in our system are the following:

- $\rho = 0.05$ kg/m (linear density of the belt);
- $\alpha = 120° = 2.094$ rad (winding angle on the motor pulley);
- $\mu = 0.2$ (friction coefficient);
- $T_1 = 120$ N (tension in the passive side of the belt);
- $T_2 = 140$ N (tension in the active side of the belt).

By using equation (10) it's possible to graph the specific pressure of the belt on the motor pulley at 15000 rpm.

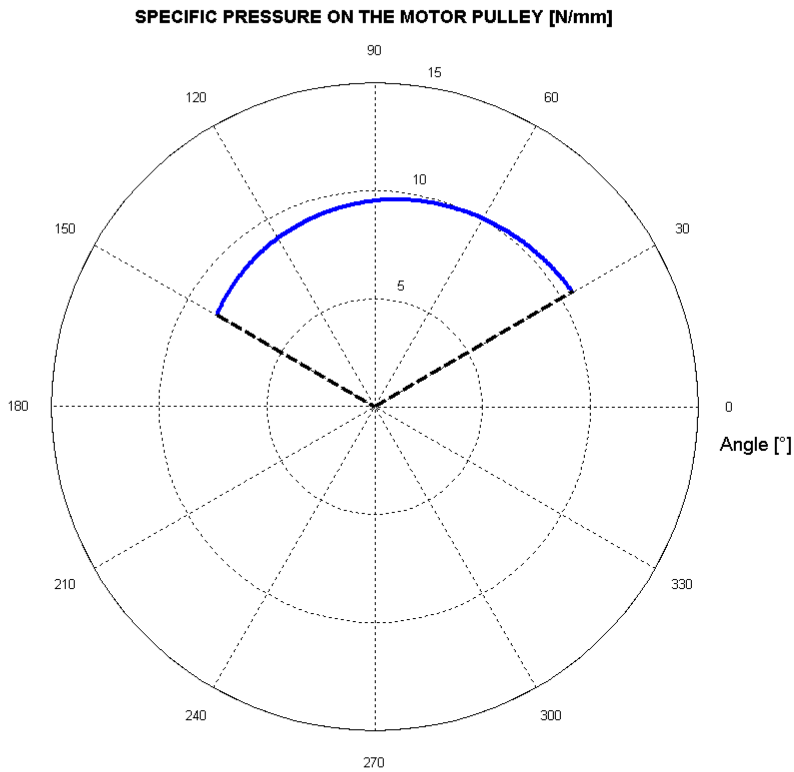


Figure 40

In figure 40 it's possible to note that the distribution of the specific pressure is always positive in its value. Therefore, even if the spinning speed is very high the separation doesn't occur. The centrifugal force is not the cause of the delay in spinning speed between motor and driven rotor.
If we analyze experimentally the behavior of the system, it's possible to note a delay at different spinning speeds at which speeds it's possible to verify, by using a stroboscope, a vibration in the active or in the passive side of the belt.
In the system, the driven pulley is slightly out of axis, and it acts as an exciting source for the belt. When an harmonic of the frequency of the driven pulley is close to a natural frequency of the belt, this starts to vibrate.
In the following table spinning speeds for the driven rotor are reported with a short description of the belt behavior.

Rotor spinning speed [rpm]	***Belt behavior***
730	Low vibrations
810	Torsion of the passive side plus transverse vibration of the active
910	Transverse vibration of the active side
1020	Transverse vibration of the active side
1090	Not relevant vibration
1180	Transverse vibration of the active side
1271	Not relevant vibration
1400	Transverse vibration of the active side
1720	Transverse vibration of the active side

Table 1

Data in the table are obtained by direct observation. In figure 42, the diagram shows as the difference in spinning speed between motor and user, changes with the frequency of rotation. In figure 43, the submultiples of frequencies of the first three modal shapes of the active side of the belt are reported. This result comes out from the equation (39).
To find submultiples we impose, by using angular velocity and radius of the user pulley,

$$V = \omega_U R_U \qquad (50)$$

and by using equation (39), it's possible to write

$$\omega_n = \frac{n\pi}{L} V_{CR}\left(1 - \frac{\omega_U^{\,2} R_U^{\,2}}{V_{CR}^{\,2}}\right). \tag{51}$$

In the figure 41, equation (51) is plotted for n=1, 2, 3. In the figure, the value of the critical frequency corresponds to the critical velocity:

$$V_{CR} = \sqrt{\frac{T}{\rho}}\,, \tag{52}$$

$$\omega_{CR} = \frac{1}{R_U}\sqrt{\frac{T}{\rho}}\,. \tag{53}$$

At this frequency, eigenvalues of the equation (24) are null. This is a condition of instability but it doesn't interest our problem.
For the equation (51), natural frequencies decrease with rotation frequency of the rotor. If some harmonics of the base frequency of the rotor are equal to a natural frequency of the belt, this will be excited as much as the order of the exciting harmonic is low. This means more energy enter in the system. Therefore, to find this frequencies of rotation, we have to satisfy the condition

$$\omega_U = \frac{\omega_n}{m} = \frac{n}{m}\frac{\pi}{L} V_{CR}\left(1 - \frac{\omega_U^{\,2} R_U^{\,2}}{V_{CR}^{\,2}}\right), \tag{54}$$

with $m \in N$. By solving the previous equation we obtain, for the exciting frequencies of rotation,

$$\omega_U = \frac{V_{CR}}{2n\pi\, R_U^{\,2}}\left(\sqrt{(mL)^2 + (2n\pi\, R_U)^2} - mL\right), \tag{55}$$

Intersection points between each dashed line and the others, in figure 41, give the values that is possible to calculate by the equation (55).
It's possible to see as the maximum delays in spinning speed corresponds to spinning speeds that are submultiples of the natural frequencies of the belt, mainly for the first, second and third modal shapes. Figure 43 can justify the phenomenon and some differences are acceptable being the result of the calculation an approximation of the behavior of the belt. In fact, to have a more correct model, the belt should be seen as a membrane, but its formulation would be uselessly complex. In table 2 a comparison between numerical and experimental values is shown. The values are the exciting rotation speeds expressed in rpm for the first three modes of the belt. *m* is the order of the harmonics. As conclusion, the transverse vibration of the belt probably reduces the winding angle and the specific pressure between belt and pulley and a slip of the belt, with a consequent delay between the two pulleys, appears.

Measured speed [rpm]	***n = 1 [rpm]***		***n = 2 [rpm]***		***n = 3 [rpm]***	
730	735	m = 8				
810	833	m = 7				
910	960	m = 6				
1020						
1180	1130	m = 5	1236	m = 9		
1400	1364	m = 4	1364	m = 8		
			1518	m = 7		
1720	1706	m = 3	1706	m = 6	1706	m = 9

Table 2

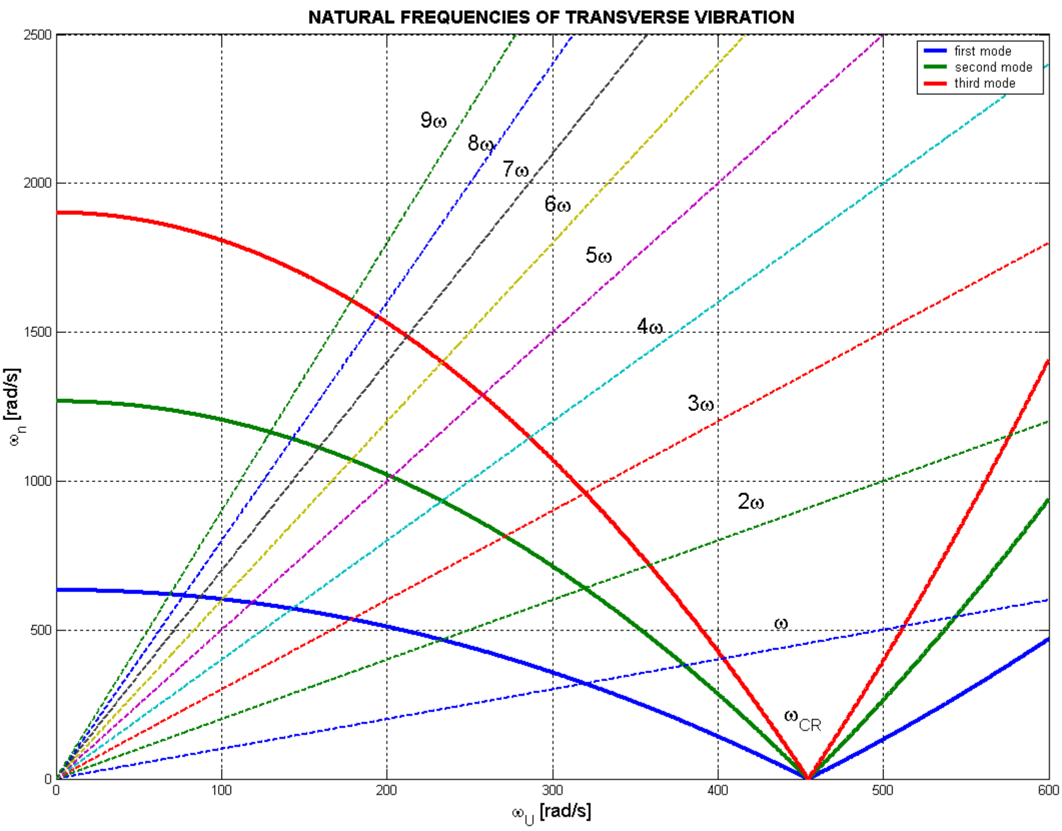
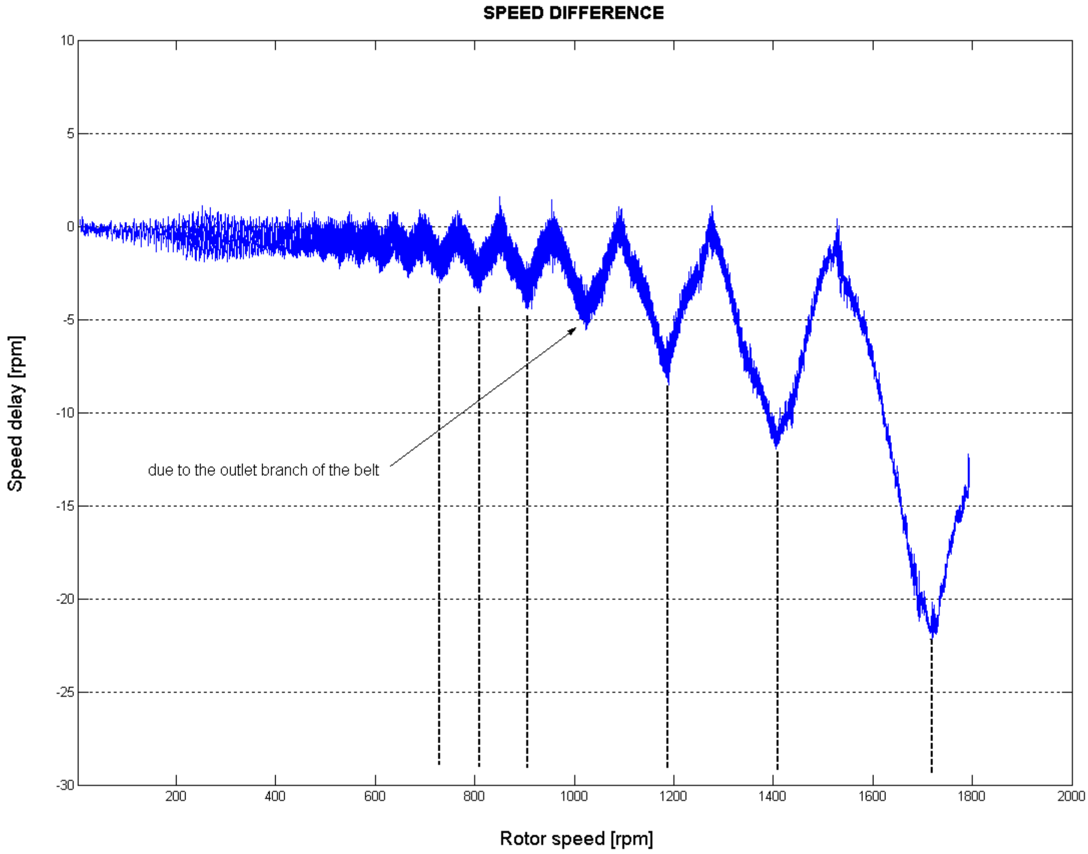


Figure 41



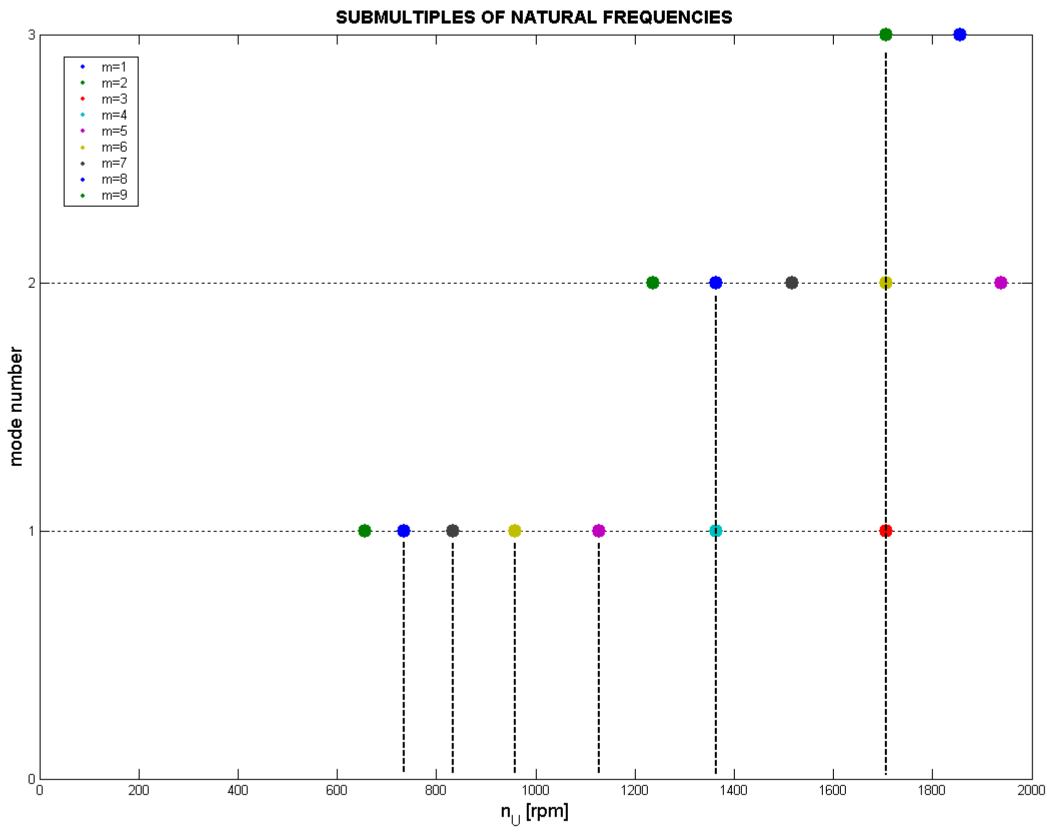


Figure 42 and 43